discussed by Collatz,[10] an equation of the form

$$a_1 - a_2 + a_3 + a_4 - a_5 - \lambda a_6 = 0 \tag{13}$$

where

$$a_1 = \frac{1}{L^2}\int_0^1 [1-\alpha(1-\xi)][b^2(0)(1-\delta\xi)^3 + t_1^2(1-\delta\xi)]\frac{d^4 f_n}{d\xi^4} f_n d\xi$$

$$a_2 = \frac{1}{L^2}\int_0^1 \langle 2\delta[1-\alpha(1-\xi)][3b^2(0)(1-\delta\xi)^2 + t_1^2] - 2\alpha[b^2(0)(1-\delta\xi)^3 + t_1^2(1-\delta\xi)]\rangle \frac{d^3 f_n}{d\xi^3} f_n d\xi$$

$$a_3 = \frac{\delta}{L^2}\int_0^1 \langle \delta[1-\alpha(1-\xi)][6b^2(0)(1-\delta\xi)] - 2\alpha[3b^2(0)(1-\delta\xi)^2 + t_1^2]\rangle \frac{d^2 f_n}{d\xi^2} f_n d\xi - \frac{\Omega^2}{\beta}\int_0^1 \left\langle \frac{r}{L}(1-\xi) + \left(1-\delta\frac{r}{L}\right)\left(\frac{1-\xi^2}{2}\right) - \frac{\delta}{3}(1-\xi^3)\right\rangle \frac{d^2 f_n}{d\xi^2} f_n d\xi$$

$$a_4 = \frac{\Omega^2}{\beta}\int_0^1 \left\langle \frac{r}{L} + \left(1-\delta\frac{r}{L}\right)\xi - \delta\xi^2\right\rangle \frac{d f_n}{d\xi} f_n d\xi$$

$$a_5 = \frac{\Omega^2}{\beta}\sin^2\Psi\int_0^1 (1-\delta\xi) f_n^2 d\xi$$

$$a_6 = \int_0^1 (1-\delta\xi) f_n^2 d\xi \tag{14}$$

The solution thus obtained would be an upper bound for the frequency parameter λ corresponding to the nth mode of vibration.

Results and Conclusions

The frequency parameter corresponding to first three modes of vibrations for various values of α and three different values of cross section variation (i.e., $\delta = 0.25$, $= 0.50$, $= 0.75$) are plotted in Fig. 2. Here the values of r/L and Ω are taken to be constant, equal to 2.0 and 314 s^{-1}, respectively, for a setting angle $\Psi = \pi/2$. The other physical constants are taken from Carnegie[11] and Rao.[12] It is observed that the frequencies decrease with an increase in α, while the frequencies increase with the increase in δ for the first mode and decrease for the second and third modes.

In order to justify the accuracy of the method employed, the results for the smallest eigenvalues have been compared with those of Ref. 4 after obtaining the frequencies from Eq. (13) using $\alpha = 0$, $\delta = 0$, $r/L = 0$ and computing

$$\gamma^{1/2} = \sqrt{[12\rho L^4/E_1(b^2(0) + t_1^2)](\omega^2 + \Omega^2\sin^2\Psi)}$$

and

$$\bar{\Omega} = \Omega\sqrt{12\rho L^4/E_1[b^2(0) + t_1^2]}$$

Obviously, our results agree with the exact results quite well.

Further, the parameter $\bar{\Omega}$ corresponding to the rotational angular velocity $\Omega = 314\ s^{-1}$ is less than 1, the maximum error in frequency of vibrations of a rotating beam without thermal effect comes out to be 0.002% from Table 1. Also, as the frequency of vibrations of the rotating beam decreases with the increasing values of α, i.e., thermal gradient, the error will always be less than 0.002% when thermal effect is included.

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Thermal Postbuckling of Columns

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THE post-buckling behavior of columns subjected to thermal loads can be described by an eigenvalue problem. The governing matrix equation is given by

$$[K]\{\delta\} - \lambda_{NL}[G]\{\delta\} = 0 \tag{1}$$

The assembled nonlinear elastic stiffness matrix $[K]$ is obtained from the element elastic stiffness matrices which are obtained from the strain energy with nonlinear strain-displacement relations of the finite elements into which the column is discretized. $[G]$ is the assembled geometric stiffness matrix obtained from the work done by the thermal loads on each of the finite elements. Cubic polynomials in the axial coordinate are assumed for the axial and transverse displacements in evaluating the element stiffness and geometric stiffness matrices. $\{\delta\}$ is the eigenvector and λ_{NL}

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Table 1 Thermal load parameter for simply supported column

a/r	γ Finite element solution Four elements[a]	Eight elements[a]	Rayleigh-Ritz solution
0.0	1.0000	1.0000	1.0000
0.2	1.0100(3)	1.0100(3)	1.0100
0.4	1.0399(5)	1.0400(5)	1.0400
0.6	1.0898(5)	1.0900(5)	1.0900
0.8	1.1598(6)	1.1600(7)	1.1600
1.0	1.2496(7)	1.2500(8)	1.2500
λ_L	9.8746	9.8699	9.8696
$\bar{a}$	0.2496	0.2500	0.25

[a]Numbers in the parentheses indicate the number of iterations required to achieve an accuracy of 10^{-4} of the converged solution.

Table 2 Thermal load parameter for clamped and simply supported/clamped columns

a/r	γ Clamped Four elements[a]	Clamped Eight elements[a]	Simply supported/clamped Four elements[a]	Simply supported/clamped Eight elements[a]
0.0	1.0000	1.0000	1.0000	1.0000
0.2	1.0024(3)	1.0025(3)	1.0059(4)	1.0059(4)
0.4	1.0098(4)	1.0100(5)	1.0236(6)	1.0237(5)
0.6	1.0221(5)	1.0225(6)	1.0530(6)	1.0533(7)
0.8	1.0392(5)	1.0399(5)	1.0942(7)	1.0948(9)
1.0	1.0613(5)	1.0624(5)	1.1472(9)	1.1480(9)
λ_L	39.7753	39.4985	20.2322	20.1934
$\bar{a}$	0.0613	0.0624	0.1472	0.1480

[a]Numbers in the parentheses indicate the number of iterations required to achieve an accuracy of 10^{-4} of the converged solution.

$[=(\alpha TL^2/r^2)_{NL}$, α being the coefficient of thermal expansion, T the rise in temperature over the stress-free state, L the length of the column, and r the radius of gyration] is the eigenvalue giving the thermal load parameter in the nonlinear range.

As the nonlinear elastic stiffness matrix $[K]$ contains the unknown displacements and their derivatives (because of nonlinear strain-displacement relations considered), an iterative method is adopted to evaluate the matrix $[K]$. Starting with the iterative procedure, the nonlinear terms in $[K]$ are taken to be zero and the solution of Eq. (1) giving the linear thermal buckling load parameter λ_L $[=(\alpha TL^2/r^2)_L]$ is obtained. The linear eigenvector $\{\delta\}_L$ is then suitably scaled up by a scalar a which is the central lateral displacement of the column and is used to obtain the nonlinear elastic stiffness matrix $[K]$. Equation (1) is solved with the new matrix $[K]$ to give λ_{NL} corresponding to the central deflection a.

Using the above formulation and the iterative method, the linear thermal buckling parameter λ_L and the thermal load parameter λ_{NL} in the nonlinear range (for various values of a/r) are calculated for simply supported, clamped, and simply supported/clamped columns. Also, an empirical formula for γ $(=\lambda_{NL}/\lambda_L)$ in the form

$$\gamma = 1+\bar{a}(a/r)^2 \tag{2}$$

where $\bar{a}$ is a constant, is evaluated using the least-squares method from the values of γ obtained for various a/r values. Table 1 gives the values of λ_L and γ (for a/r ranging between 0.0 and 1.0 in increments of 0.2) for a simply supported column for four- and eight-element idealizations (of the full column). The $\bar{a}$ value of the empirical formula for γ is also given.

In order to establish the accuracy of the finite element formulation described above, a Rayleigh-Ritz analysis is carried out for the case of simply supported columns. The usual displacement distributions for the axial and transverse displacements, in terms of trigonometric functions, are assumed, which satisfy the geometric boundary conditions. Using these displacement distributions and minimizing the total potential energy, a system of nonlinear algebraic equations is obtained which yields the solution for λ_{NL}/λ_L as

$$\gamma = \frac{\lambda_{NL}}{\lambda_L} = 1+\frac{1}{4}\left(\frac{a}{r}\right)^2 \tag{3}$$

where $\lambda_L = \pi^2$.

In view of the assumed displacement distributions for simply supported columns, this solution is considered to be exact. A comparison of the finite element results (Table 1) for eight-element idealization with these results shows an excellent agreement and thus establishes the accuracy of the finite element scheme employed.

The results for clamped and simply supported clamped columns are presented in Table 2 for both four- and eight-element idealization (in full columns). The convergence of the results can be seen to be very good and as such an eight-element solution can be taken to be very accurate.

The difficulty in choosing appropriate displacement distributions for the Rayleigh-Ritz analysis can be overcome by this alternate simple finite element formulation. The results show that the effect of nonlinearity on the load ratios decreases from simply supported to simply supported/clamped and then to clamped columns.

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Vibration of Orthotropic Thick Circular Plates

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Introduction

Large amplitude vibration studies of circular plates have been carried out by Nowinski,[1] Yamaki,[2] Wah,[3] and several others.[4] While most of the investigations are concerned with isotropic plates, some deal with orthotropic circular plates as well. Classical nonlinear thin plate theory

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